Note Title 10/27/2007

Open loop transfer function : $H(s)$

$$H(s) = \frac{A}{(1+\tau_1 s)(1+\tau_2 s)(1+\tau_3 s)}$$

where $A = 1e7$

$$\frac{1}{\tau_1} = 159.15\,Hz \times 2\pi$$

$$\frac{1}{\tau_2} = 100\,KHz \times 2\pi$$

$$\frac{1}{\tau_3} = 100\,MHz \times 2\pi$$

In unity gain feedback the transfer function becomes :

$$H_F(s) = \frac{H(s)}{1+H(s)} = \frac{A}{1 + A + s(\tau_1+\tau_2+\tau_3) + s^2(\tau_1\tau_2 + \tau_2\tau_3 + \tau_1\tau_3) + s^3\,\tau_1\tau_2\tau_3}$$

or $$H_F(s) = \frac{A/A+1}{1 + \frac{s(\tau_1+\tau_2+\tau_3)}{A+1} + \frac{s^2(\tau_1\tau_2+\tau_2\tau_3+\tau_1\tau_3)}{A+1} + \frac{s^3\tau_1\tau_2\tau_3}{A+1}}$$

Setting $B = \frac{A}{A+1}$ & the roots of the denominator be $P_1, P_2, P_3$

$$\therefore H_F(s) = \frac{B}{(1+T_1 s)(1+T_2 s)(1+T_3 s)}$$

where $T_1 = \frac{-1}{P_1}$, $T_2 = \frac{-1}{P_2}$, $T_3 = -\frac{1}{P_3}$

Given the input as a step the output is given as $R(s)$

$$R(s) = \frac{B}{s(1+T_1 s)(1+T_2 s)(1+T_3 s)}$$

$$= \frac{K_1}{1+T_1 s} + \frac{K_2}{1+T_2 s} + \frac{K_3}{1+T_3 s} + \frac{K_4}{s}$$

$$B = K_1 s(1+T_2 s)(1+T_3 s) + K_2(1+T_1 s)(1+T_3 s)s + K_3(1+T_1 s)(1+T_2 s)s + K_4(1+T_1 s)(1+T_2 s)(1+T_3 s)$$

Set $s = 0$ then $B = K_4$

Set $s = -\frac{1}{T_1}$ then $B = -\frac{K_1}{T_1}\left(1 - \frac{T_2}{T_1}\right)\left(1 - \frac{T_3}{T_1}\right)$

$$\text{or } \quad K_1 = \frac{-B T_1}{\left(1 - \frac{T_2}{T_1}\right)\left(1 - \frac{T_3}{T_1}\right)}$$

Similarly $$K_2 = \frac{-B T_2}{\left(1 - \frac{T_1}{T_2}\right)\left(1 - \frac{T_3}{T_2}\right)}$$

$$K_3 = \frac{-B T_3}{\left(1 - \frac{T_1}{T_3}\right)\left(1 - \frac{T_2}{T_3}\right)}$$

$$\therefore R(s) = \frac{K_1}{1+T_1 s} + \frac{K_2}{1+T_2 s} + \frac{K_3}{1+T_3 s} + \frac{B}{s}$$

Note
$$\frac{1}{s+a} \xleftrightarrow{\mathcal{L}} e^{-at}$$

$$\frac{1/a}{s/a+1} \xleftrightarrow{\mathcal{L}} e^{-at}$$

$$\frac{\tau}{\tau s+1} \xleftrightarrow{\mathcal{L}} e^{-t/\tau}$$

$$\text{or } \quad \frac{1}{\tau s+1} \xleftrightarrow{\mathcal{L}} \frac{1}{\tau} e^{-t/\tau}$$

$$\therefore\ r(t) = \frac{K_1}{T_1} e^{-t/\tau_1} + \frac{K_2}{T_2} e^{-t/\tau_2} + \frac{K_3}{T_3} e^{-t/\tau_3} + B u(t)$$

$$r(t) = \underbrace{\frac{-B}{\left(1-\frac{T_2}{T_1}\right)\left(1-\frac{T_3}{T_1}\right)}}_{I} e^{-t/\tau_1} - \underbrace{\frac{B}{\left(1-\frac{T_1}{T_2}\right)\left(1-\frac{T_3}{T_2}\right)}}_{II} e^{-t/\tau_2} - \underbrace{\frac{B}{\left(1-\frac{T_1}{T_3}\right)\left(1-\frac{T_2}{T_3}\right)}}_{III} e^{-t/\tau_3}$$

$$+ \underbrace{\frac{A}{1+A}}_{IV} u(t)$$

$$r(t) = \frac{-B}{\left(1-\frac{T_2}{T_1}\right)\left(1-\frac{T_3}{T_1}\right)} e^{P_1 t} - \frac{B}{\left(1-\frac{T_1}{T_2}\right)\left(1-\frac{T_3}{T_2}\right)} e^{P_2 t} - \frac{B}{\left(1-\frac{T_1}{T_3}\right)\left(1-\frac{T_2}{T_3}\right)} e^{P_3 t} + \frac{A}{1+A} u(t)$$

We can solve for $P_1, P_2, P_3$ (roots of denominator of $H_F(s)$)

$$P_1 = -\frac{1}{T_1} = -6.3803 \times 10^8 \text{ rad/sec} = a_1$$

$$P_2 = -\frac{1}{T_2} = (0.0454 + 0.7853j) \times 10^8 \text{ rad/sec} = a_2 + jb_2$$

$$P_3 = -\frac{1}{T_3} = (0.0454 - 0.7853j) \times 10^8 \text{ rad/sec} = a_2 - jb_2$$

I :

$$1 - \frac{T_2}{T_1} = 1 - \frac{a_1}{a_2 + jb_2} = \frac{(a_2 - a_1) + jb_2}{a_2 + jb_2}$$

$$1 - \frac{T_3}{T_1} = 1 - \frac{a_1}{a_2 - jb_2} = \frac{(a_2 - a_1) - jb_2}{a_2 - jb_2}$$

$$\left(1 - \frac{T_2}{T_1}\right)\left(1 - \frac{T_3}{T_1}\right) = \frac{(a_2 - a_1)^2 + b_2^2}{a_2^2 + b_2^2}$$

$\therefore$ I : $\dfrac{-B(a_2^2+b_2^2)}{(a_2-a_1)^2+b_2^2}e^{a_1t}$

II & III terms (Denominators) :

$$\left(1-\frac{T_1}{T_2}\right)\left(1-\frac{T_3}{T_2}\right)=\left(1-\frac{a_2+jb_2}{a_1}\right)\left(1-\frac{a_2+jb_2}{a_2-jb_2}\right)=\left(\frac{(a_1-a_2)-jb_2}{a_1}\right)\left(\frac{-2jb_2}{a_2-jb_2}\right)$$

$$\left(1-\frac{T_1}{T_3}\right)\left(1-\frac{T_2}{T_3}\right)=\left(1-\frac{a_2-jb_2}{a_1}\right)\left(1-\frac{a_2-jb_2}{a_2+jb_2}\right)=\left(\frac{(a_1-a_2)+jb_2}{a_1}\right)\left(\frac{2jb_2}{a_2+jb_2}\right)$$

II & III terms :

$$\frac{-Ba_1}{2jb_2}\left[-\frac{(a_2-jb_2)}{(a_1-a_2)-jb_2}e^{a_2t}e^{jb_2t}+\frac{a_2+jb_2}{(a_1-a_2)+jb_2}e^{a_2t}e^{-jb_2t}\right]$$

$$=\frac{-Ba_1e^{a_2t}}{2jb_2}\left[-\underbrace{\frac{(a_2-jb_2)}{(a_1-a_2)-jb_2}e^{jb_2t}}_{x-jy}+\underbrace{\frac{a_2+jb_2}{(a_1-a_2)+jb_2}e^{-jb_2t}}_{x+jy}\right]$$

$$x+jy-(x-jy)=2jy.$$

$\therefore$ II & III terms :

$$=-\frac{Ba_1e^{a_2t}}{2jb_2}2jy=-\frac{Ba_1e^{a_2t}y}{b_2}$$

Here $y=\text{Imaginary}\left[\dfrac{a_2+jb_2}{(a_1-a_2)+jb_2}e^{-jb_2t}\right]$

$$y = \text{Imag}\left[\frac{(a_2+jb_2)((a_1-a_2)-jb_2)}{(a_1-a_2)^2+b_2^2}\left[\cos b_2 t - j\sin b_2 t\right]\right]$$

$$= \underbrace{\frac{1}{(a_1-a_2)^2+b_2^2}}_{M_3}\left[\underbrace{\left[(a_1-a_2)b_2 - a_2 b_2\right]}_{M_1}\cos b_2 t - \underbrace{\left[a_2(a_1-a_2)+b_2^2\right]}_{M_2}\sin b_2 t\right]$$

$$= \frac{1}{M_3}\left[M_1\cos b_2 t - M_2 \sin b_2 t\right]$$

$$= \frac{\sqrt{M_1^2+M_2^2}}{M_3}\left[\frac{M_1}{\sqrt{M_1^2+M_2^2}}\cos b_2 t - \frac{M_2}{\sqrt{M_1^2+M_2^2}}\sin b_2 t\right] \qquad \text{Let } \tan\theta = \frac{M_1}{M_2}$$

$$= \frac{\sqrt{M_1^2+M_2^2}}{M_3}\left[\sin\theta\cos b_2 t - \cos\theta\sin b_2 t\right]$$

$$= \frac{\sqrt{M_1^2+M_2^2}}{M_3}\sin(\theta - b_2 t)$$

$\therefore$ II & III terms are:

$$= -\frac{B a_1 e^{a_2 t}}{b_2}\frac{\sqrt{M_1^2+M_2^2}}{M_3}\sin(\theta - b_2 t)$$

$$\therefore r(t) = \frac{-B(a_2^2+b_2^2)}{(a_2-a_1)^2+b_2^2}e^{a_1 t} + \frac{B a_1 e^{a_2 t}}{b_2}\frac{\sqrt{M_1^2+M_2^2}}{M_3}\sin(b_2 t - \theta)$$

$$+ \frac{A}{1+A}u(t)$$

So we see the oscillation frequency is $b_2$

$$f = \frac{b_2}{2\pi} = \frac{0.7853\times10^8}{2\pi}\text{ Hz} = 12.498\text{ MHz}$$